Monday Night Calculus, December 6, 2021

1. Consider the curve described by $2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$. (Grace Jang)

(a) Find $\dfrac{dy}{dx}$.

$$6y^2y' + 2yy' - 5y^4y' = 4x^3 - 6x^2 + 2x$$

$$y'(6y^2 + 2y - 5y^4) = 4x^3 - 6x^2 + 2x$$

$$y' = \frac{4x^3 - 6x^2 + 2x}{6y^2 + 2y - 5y^4}$$

(b) Find the points on the curve at which the tangent line is vertical.

$$-5y^4 + 6y^2 + 2y = 0 \quad \Rightarrow \quad y(-5y^3 + 6y + 2) = 0$$

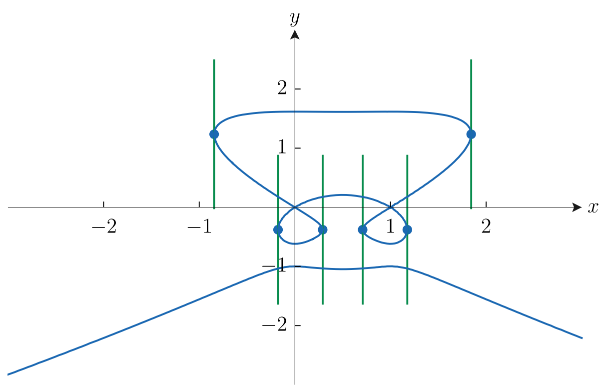
$$y = 0, \quad -0.856, \quad -0.379, \quad 1.235$$

$$y = 0 \quad \Rightarrow \quad x = 0,\ 1 \quad \Rightarrow \quad 4 \cdot 0^3 - 6 \cdot 0^2 + 2 \cdot 0 = 0,\ 4 \cdot 1^3 - 6 \cdot 1^2 + 2 \cdot 1 = 0\ ??$$

$$y = -0.856 \quad \Rightarrow \quad x = \text{ complex}$$

$$y = -0.379 \quad \Rightarrow \quad x = -0.176, \quad 0.291, \quad 0.709, \quad 1.176$$

$$y = 1.235 \quad \Rightarrow \quad x = -0.844, \quad 1.844$$



2. The rate at which rainwater flows into a drainpipe is modeled by the function R where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out of the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$. Determine whether each statement is True or False.

(Patty Odette Jacobs)

(a) The amount of water in the pipe is increasing at $t = 3$.

$R(3) - D(3) = -0.314 < 0 \quad \Rightarrow \quad$ water in the pipe is decreasing at time $t = 3$ hours.

(b) The rate at which water is draining from the pipe at $t = 1$ is 1.64 cubic feet per hour.

$D(1) = 1.32$

(c) The rate at which water is draining from the pipe at $t = 1$ is 1.32 cubic feet per hour.

$D(1) = 1.32$

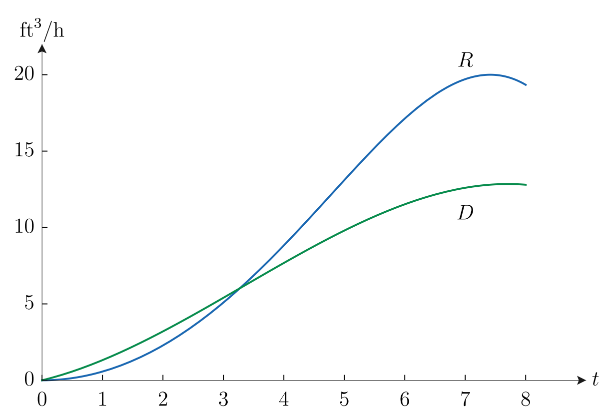
(d) The rate at which water is draining from the pipe at $t = 1$ is increasing at a rate of 1.64 cubic feet per hour per hour.

$D'(1) = 1.64$

(e) The volume of water in the pipe at $t = 4$ is decreasing at a rate of 1.148 cubic feet per hour.

$R(4) - D(4) = 1.148$

(f) The volume of water in the pipe has zero rate of change for some time t on $[2, 5]$.



$R(t) - D(t) = 0 \quad \Rightarrow \quad t = 3.272$ hours

(g) The rate at which rainwater flows into the pipe and the rate at which the rainwater flows out of the pipe is the same at time $t = 0$.

$R(0) = D(0) = 0$

(h) At time $t = 2$ more water is draining from the pipe than is flowing into the pipe.

$R(2) - D(2) = -0.919 < 0$: more water draining from the pipe.

(i) The rate of change of volume in the tank is given by $30 + R(t) - D(t)$.

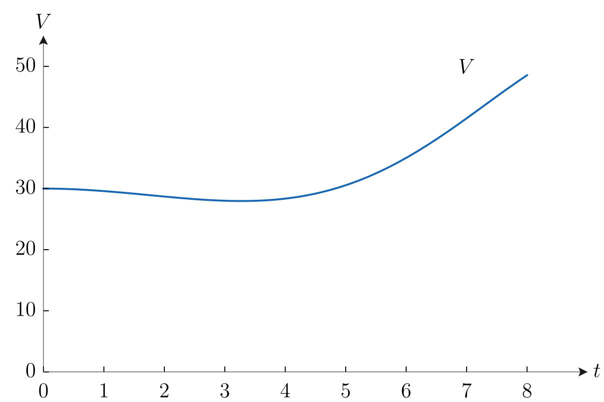
The rate of change should not include the constant term 30.

(j) Find the maximum amount of water in the pipe over the time interval $[0, 8]$.

$$V(t) = 30 + \int_0^t (R(x) - D(x))\,dx$$

$$V'(t) = R(t) - D(t) = 0 \quad \Rightarrow \quad t = 0,\ 3.272$$

t	$V(t)$
0	30
3.272	27.967
8	48.544



3. Let g be the function given by $g(x) = \sqrt{1 - \sin^2 x}$. Which of the following statements could be false on the interval $0 \le x \le \pi$?

(A) By the Extreme Value Theorem, there is a value c such that $g(c) \le g(x)$ for $0 \le x \le \pi$.

(B) By the Extreme Value Theorem, there is a value c such that $g(c) \ge g(x)$ for $0 \le x \le \pi$.

(C) By the Intermediate Value Theorem, there is a value c such that $g(c) = \dfrac{g(0) + g(\pi)}{2}$.

(D) By the Mean Value Theorem, there is a value c such that $g'(c) = \dfrac{g(\pi) - g(0)}{\pi - 0}$.

Extreme Value Theorem (EVT)

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The Intermediate Value Theorem (IVT)

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. There there exists a number c in (a, b) such that $f(c) = N$.

The Mean Value Theorem (MVT)

Let f be a function that satisfies the following hypotheses:

(1) f is continuous on the closed interval $[a, b]$.

(2) f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

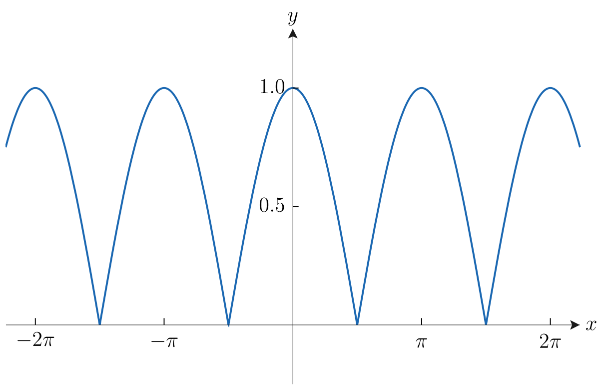
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalently, $f(b) - f(a) = f'(c)(b - a)$

$g(x) = \sqrt{1 - \sin^2 x} = \sqrt{\cos^2 x} = |\cos x|$

Domain: $(-\infty, \infty)$ g is continuous on its domain.

$$g'(x) = \frac{1}{2}(1 - \sin^2 x)^{-1/2} \cdot -2 \sin x \cos x = -\frac{\sin x \cos x}{\sqrt{1 - \sin^2 x}}$$

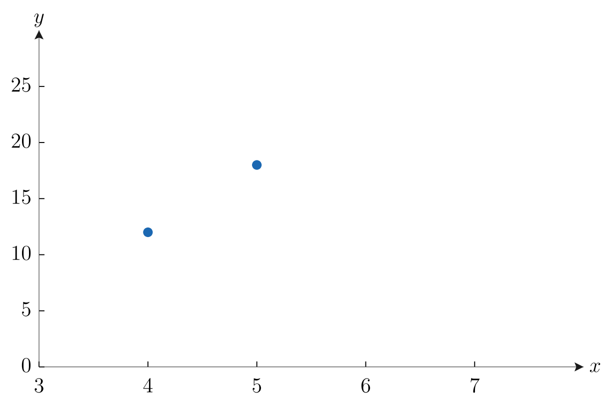


g is not differentiable at $x = \dfrac{\pi}{2} \pm n\pi$

4. Let g be a twice differentiable function with $g'(x) > 0$ and $g''(x) < 0$ for all real numbers x, such that $g(4) = 12$ and $g(5) = 18$. Of the following, which is a possible value for $g(6)$?

(Judy Mitchell Barnette)

(A) 15 (B) 18 (C) 21 (D) 24



5. $\int_1^9 t(3t^2-1)^5\,dt$ (Laurel Kerg)

$$u = 3t^2 - 1 \qquad t = 1:\ u = 3\cdot 1^2 - 1 = 2$$
$$du = 6t\,dt \qquad t = 9:\ u = 3\cdot 9^2 - 1 = 242$$
$$dt = \frac{du}{6t}$$

$$\int_1^9 t(3t^2-1)^5\,dt = \int_2^{242} t(u)^5\,\frac{du}{6t}$$ Change variables.

$$= \frac{1}{6}\int_2^{242} u^5\,du$$ Simplify.

$$= \frac{1}{6}\left[\frac{u^6}{6}\right]_2^{242}$$ Antiderivative.

$$= \frac{1}{36}(242^6 - 2^6) = 5579428225280$$ FTC.

6. $\int_{-\infty}^{0} 6e^{2x}\,dx$ (Mary Loose)

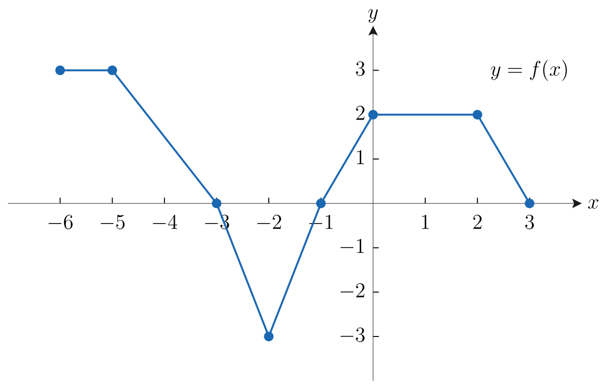
$$\int_{-\infty}^{0} 6e^{2x}\,dx = \lim_{t\to-\infty} \int_{t}^{0} 6e^{2x}\,dx$$ Improper Integral.

$$= \lim_{t\to-\infty} \left[\frac{6}{2}e^{2x}\right]_t^0$$ Antiderivative.

$$= \lim_{t\to-\infty} 3[e^0 - e^t]$$ FTC.

$$= 3\lim_{t\to-\infty}(1-0) = 3$$ Evaluate the limit.

7. The graph of the function f is shown in the figure. (Ryan Waggoner)



The function h is defined by $h(x) = \int_{-1}^{x} f(t)\,dt$ for $-6 \le x \le 3$.

(a) Find the values at which h has a relative extreme value.

(b) Find the values at which h has an absolute extreme value.

(c) Find the values at which the graph of h has an inflection point.

(d) Find the intervals on which h is increasing.

Solution

(a) $h'(x) = f(x)$

$h'(x) = f(x) = 0 : x = -3,\ -1$

$g'(x) = f(x)$ DNE : none

h has a relative maximum at $x = -3$ because $h'(x)$ changes from positive to negative there.

h has a relative minimum at $x = -1$ because $h'(x)$ changes from negative to positive there.

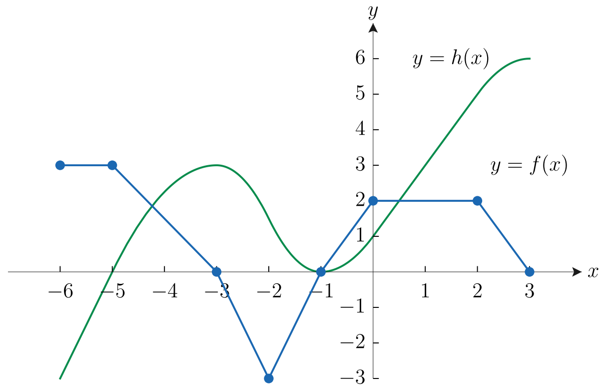
(b)

x	$h(x)$
-6	-3
-3	3
-1	0
3	6

(c) $h''(x) = f'(x)$

h has an inflection point at $x = -2$ because $h' = f$ changes from decreasing to increasing there.

(d) h increasing: $[-6, -3],\ [-1, 3]$



y
x
y = h(x)
y = f(x)
6
5
4
3
2
1
−1
−2
−3
−6
−5
−4
−3
−2
−1
1
2
3